Math 115A - All Theorems and Definitions

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Please note that I didn't make any effort to distinguish the zero vector and the zero scalar throughout this document

1 Vector Spaces

1.1 Introduction

Theorem 1.0. *Nobody cares about this section.*

1.2 Vector Spaces

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Definition. *A* ***vector space*** *V over a field F consists of a set on which two operations (called* ***addition*** *and* ***scalar multiplication****, respectively) are defined so that for each pair of elements x, y, in V there is a unique element x + y in V, and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold:*

1. (VS 1) For all x, y in V, x + y = y + x (commutativity of addition).
2. (VS 2) For all x, y, z in V, (x + y) = z = x + (y + z) (associativity of addition)
3. (VS 3) There exists an element in V denoted by 0 such that x + 0 = x for each x in V.
4. (VS 4) For each element x in V there exists an element y in V such that x + y = 0. (y = -x)
5. (VS 5) For each element x in V, 1x = x.
6. (VS 6) For each pair of elements a, b in F, and each element x in V, (ab)x = a(bx).
7. (VS 7) For each element a in F and each pair of elements x, y in V, a(x + y) = ax + ay.
8. (VS 8) For each pair of elements a, b in F and each element x in V, (a + b)x = ax + bx

Theorem 1.1 (Cancellation Law for Vector Addition). *If x, y, z are vectors in a vector space V such that x + z = y + z, then x = y.*

Corollary. *The vector 0 described in (VS 3) is unique.*

Corollary. *The vector y=-x described in (VS 4) is unique.*

Theorem 1.2. *In any vector space V, the following statements are true:*

1. 0x = 0 for each x ∈ V.
2. (-a)x = -(ax) = a(-x) for each a ∈ F and each x ∈ V
3. a0 = 0 for each a ∈ F.

1.3 Subspaces

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Definition. *A subset W of a vector space V over a field F is called a* ***subspace*** *of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.*

Theorem 1.3. *Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the oprations defined in V.*

1. 0_V ∈ W.
2. x + y ∈ W whenever x ∈ W and y ∈ W.
3. cx ∈ W whenever c ∈ F and x ∈ W.

Theorem 1.4. *Any intersection of subspaces of a vector space V is a subspace of V.*

1.4 Linear Combinations and Systems of Linear Equations

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Definition. *Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a* ***linear combination*** *of vectors of S if there exist a finite number of vectors $u_1, u_2, ..., u_n$ in S and scalars $a_1, a_2, ..., a_n$ in F such that $v = a_1u_1+a_2u_2+...+a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2, ..., u_n$ and call $a_1, a_2, ..., a_n$ the* ***coefficients*** *of the linear combination.*

Definition. *Let S be a nonempty subset of a vector space V. The* **span** *of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span(0) = {0}.*

Theorem 1.5. *The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.*

Definition. *A subset S of a vector space V* ***generates*** *(or* **spans***) V if span(S) = V. In this case, we also say that the vectors of S generate (or span) V.*

1.5 Linear Dependence and Linear Independence

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Definition. *A subset S of a vector space V is called* ***linearly dependent*** *if there exist a finite number of distinct vectors $u_1, u_2, ..., u_n$ in S and scalars $a_1, a_2, ..., a_n$, not all zero, such that*

$$a_1u_1 + a_2v_2 + ... + a_nv_n = 0 \tag{1}$$

For any vectors $u_1, u_2, ..., u_n$, we have $a_1u_1 + a_2u_2 + ... + a_nu_n = 0$ if $a_1 = a_2 = ... = a_n = 0$. We call this the ***trivial representation*** *of 0 as a linear combination of $u_1, u_2, ...u_n$.*

Definition. *A subset S of a vector space that is not linearly dependent is called* ***linearly independent****. As before, we also say that the vectors of S are linearly independent.*

Theorem 1.6. *Let V be a vector space, and let $S_1 \subset S_2 \subset V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*

Corollary. *Let V be a vector space, and let $S_1 \subset S_2 \subset V$. If S_2 is linearly independent, then S_1 is linearly independent.*

Theorem 1.7. *Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in$ span(S).*

1.6 Bases and Dimension

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Definition. *A* ***basis*** *β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.*

Theorem 1.8. *Let V be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form*

$$v = a_1v_1 + a_2v_2 + ... + a_nu_n \tag{2}$$

for unique scalars $a_1, a2, ..., a_n$.

Theorem 1.9. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Theorem 1.10 (Replacement Theorem). *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then* $m \leq n$ *and there exists a subset H of G containing exactly n - m vectors such that* $L \cup H$ *generates V.*

Corollary. *Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.*

Definition. *A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by dim(V). A vector space that is not finite-dimensional is called **infinite-dimensional**.*

Corollary. *Let V be a vector space with dimension n. Then*

1. Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.
2. Any linearly independent subset of V that contains exactly n vectors is a basis for V.
3. Every linearly independent subset of V can be extended to a basis for V.

Theorem 1.11. *Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and* $dim(W) \leq dim(V)$*. Moreover, if dim(W) = dim(V), then V = W.*

Corollary. *If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.*

2 Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

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Definition. *Let V and W be vector spaces (over F). We call a function T:* $V \to W$ *a **linear transformation** from V to W if, for all x,* $y \in V$ *and* $c \in F$*, we have*

1. T(x + y) = T(x) + T(y) and
2. T(cx) = cT(x)

Properties.

1. If T is linear, then T(0) = 0.
2. T is linear if and only if T(cx + y) = cT(x) + T(y) for all x, y $\in$ V and c $\in$ F.
3. If T is linear, then T(x-y) = T(x) - T(y) for all x, y $\in$ V.
4. T is linear if and only if, for $x_1, x_2, ..., x_n \in V$ *and* $a_1, a_2, ..., a_n \in$ F, we have

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i). \tag{3}$$

We generally use property 2 to prove that a given transformation is linear.

Definition. *Let V and W be vector spaces, and let T:* $V \to W$ *be linear. We define the **null space** (or **kernel**) N(T) of T to be the set of all vectors x in V such that T(x) =* ***0****; that is,* $N(T) = \{x \in V : T(x) = \mathbf{0}\}$*.*
*We define the **range** (or **image**) R(T) of T to be the subset of W consisting of all images (under T) of vectors in V; that is,* $R(T) = \{T(x) : x \in V\}$*.*

Theorem 2.1. *Let V and W be vector spaces and T: $V \to W$ be linear. Then N(T) and R(T) are subspaces of V and W, respectively.*

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Theorem 2.2. *Let V and W be vector spaces and T: $V \to W$ be linear. If $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V, then*

$$R(T) = span(T(\beta)) = span(\{T(v_1), T(v_2), ..., T(v_n)\}). \tag{4}$$

Definition. *Let V and W be vector spaces, and let T:$V \to W$ be linear.If N(T) and R(T) are finite-dimensional, then we define the* ***nullity*** *of T, denoted nullity(T), and the* ***rank*** *of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.*

Theorem 2.3 (Dimension Theorem). *Let V and W be vector spaces and T: $V \to W$ be linear. If V is finite-dimensional, then*

$$nullity(T) + rank(T) = dim(V). \tag{5}$$

Theorem 2.4. *Let V and W be vector spaces and T: $V \to W$ be linear. Then T is one-to-one if and only if N(T) = $\{\mathbf{0}\}$.*

Theorem 2.5. *Let V and W be vector spaces of equal (finite) dimension and T: $V \to W$ be linear. Then the following are equivalent.*

1. T is one-to-one.
2. T is onto.
3. rank(T) = dim(V).

Theorem 2.6. *Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, ..., v_n\}$ is a basis for V. For $\{w_1, w_2, ..., w_n\}$ in W, there exists exactly one linear transformation T: $V \to W$ such that $T(v_i) = w_i$ for i = 1, 2, ..., n.*

Corollary. *Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, ..., v_n\}$.*
If U, T: $V \to W$ are linear and $U(v_i) = T(v_i)$ for i = 1, 2, ..., n, then U = T.

2.2 The Matrix Representation of a Linear Transformation

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Definition. *Let V be a finite-dimensional vector space. An* ***ordered basis*** *for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*
For the vector space F^n, we call $\{e_1, e_2, ..., e_n\}$ the ***standard ordered basis*** *for F^n. Similarly, for the vector space $P_n(F)$, we call $\{1, x, ..., x^n\}$ the* ***standard ordered basis*** *for $P_n(F)$.*

Definition. *Let $\beta = \{u_1, u_2, ..., u_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$, let $a_1, a_2, ..., a_n$ be the unique scalars such that*

$$x = \sum_{i=1}^{n} a_i u_i. \tag{6}$$

we define the ***coordinate vector of*** *x* ***relative to*** *β, denoted $[x]_\beta$, by*

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{7}$$

Definition. *Using the notation above, we call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the* ***matrix representation of T in the ordered bases*** *β and γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$*

Notice that the jth column of A is simply $[T(v_j)]_\gamma$. Also observe that if U: V $\rightarrow$ W is a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$, then $U = T$ by the corollary to Theorem 2.6.

Definition. *Let T, U: V $\rightarrow$ W be arbitrary functions, where V and W are vector spaces over F, and let a $\in$ F. We define T + U : V $\rightarrow$ W by (T + U)(x) = T(x) + U(x) for all x $\in$ V, and aT: V $\rightarrow$ W by (aT)(x) = aT(x) for all x $\in$ V.*

Theorem 2.7. *Let V and W be vector spaces over field F, and let T, U: V $\rightarrow$ W be linear.*

1. For all a $\in$ F, aT + U is linear.
2. Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F.

Definition. *Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by L(V, W). In the case that V = W, we write L(V) instead.*

Theorem 2.8. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let T, U: V $\rightarrow$ W be linear transformations. Then*

1. $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a

2.3 Composition of Linear Transformations and Matrix Multiplication

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Theorem 2.9. *Let V, W, and Z be vector spaces over the same field F, and let T: V $\rightarrow$ W and U: W $\rightarrow$ Z be linear. Then UT: V $\rightarrow$ Z is linear.*

Theorem 2.10. *Let V be a vector space. Let T, $U_1, U_2 \in L(V)$. Then*

1. $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
2. $T(U_1U_2) = (TU_1)U2$
3. TI = IT = T
4. $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a.

Definition. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the* ***product*** *of A and B, denoted AB, to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \text{ for } 1 \leq i \leq m, 1 \leq j \leq p. \tag{8}$$

Theorem 2.11. *Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let T: V $\rightarrow$ W and U: W $\rightarrow$ Z be linear transformations. Then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta. \tag{9}$$

Corollary. *Let V be a finite-dimensional vector space with an ordered basis β. Let T, U $\in$ L(V). Then $[UT]_\beta = [U]_\beta [T]_\beta$.*

Definition. *We define the **Kronecker delta** δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ **identity matrix** I_n is defined by $(I_n)_{ij} = \delta_{ij}$. Thus, for example,*

$$I_1 = (1), I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

Theorem 2.12. *Let A be an m×n matrix, B and C be n×p matrices, and D and E be q×m matrices. Then*

1. A(B + C) = AB + AC and (D + E)A = DA + EA.
2. a(AB) = (aA)B and (D + E)A = DA + EA for any scalar a.
3. $I_m A = A = AI_n$.
4. If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.

Corollary. *Let A be an m×n matrix, $B_1, B_2, ..., B_k$ be n×p matrices, $C_1, C_2, ..., C_k$ be q×m matrices, and $a_1, a_2, ..., a_k$ be scalars. Then*

$$A(\sum_{i=1}^{k} a_i B_i) = \sum_{i=1}^{k} a_i AB_i \tag{11}$$

and

$$(\sum_{i=1}^{k} a_i C_i)A = \sum_{i=1}^{k} a_i C_i A. \tag{12}$$

Theorem 2.13. *Let A be an m×n matrix and B be an n×p matrix. For each j ($1 \leq j \leq p$) let u_j and v_j denote the jth column of AB and B, respectively. Then*

1. $u_j = Av_j$
2. $v_j = Be_j$, where e_j is the jth standard vector of F^p.

Theorem 2.14. *Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let T: V → W be linear. Then, for each u ∈ V, we have*

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta \tag{13}$$

Definition. *Let A be an m×n matrix with entries from a field F. We denote by L_A the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for eachc column vector $x \in F^n$. We call L_A a **left-multiplication transformation**.*

Theorem 2.15. *Let A be an m×n matrix with entries from F. Then the left-multiplication transformation $L_A : F^n \to F^m$ is linear. Furthermore, if B is any other m×n matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties.*

1. $[L_A]_\beta^\gamma$ = A
2. $L_A = L_B$ if and only if A = B
3. $L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all a ∈ F.
4. If $T : F^n \to F^m$ is linear, then there exists a unique m×n matrix C such that T = L_C. In fact, C = $[T]_\beta^\gamma$.
5. If E is an n×p matrix, then $L_{AE} = L_A L_E$
6. If m = n, then $L_{I_n} = I_{F^n}$

Theorem 2.16. *Let A, B, and C be matrices such that A(BC) is defined. Then (AB)C is also defined and A(BC) = (AB)C; that is, matrix multiplication is associative.*

2.4 Invertibility and Isomorphisms

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Definition. *Let V and W be vector spaces, and let T: U $\rightarrow$ V be linear. A function U: W $\rightarrow$ V is said to be an* ***inverse*** *of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be* ***invertible****. As noted in appendix B, if T is invertible, then the inverse of T is unique and is denoted by T^{-1}.*

The following facts hold for invertible functions T and U.

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$; in particular, T^{-1} is invertible.
3. Let T: V $\rightarrow$ W be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if rank(T) = dim(V).

Theorem 2.17. *Let V and W be vector spaces, and let T: V $\rightarrow$ W be linear and invertible. Then T^{-1} : $W \rightarrow V$ is linear.*

Definition. *Let A be an n×n matrix. Then A is* ***invertible*** *if there exists an n×n matrix B such that AB = BA = 1.*
If A is invertible, then the matrix B such that AB = BA = I is unique. (If C were another such matrix, then C = CI = C(AB) = (CA)B = IB = B.) The matrix B is called the ***inverse*** *of A and is denoted by A^{-1}.*

Lemma. *Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case, dim(V) = dim(W).*

Theorem 2.18. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let T: V $\rightarrow$ W be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.*

Corollary. *Let V be a finite-dimensional vector space with an ordered basis β, and let T: V $\rightarrow$ V be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.*

Corollary. *Let A be an n×n matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(L_A)^{-1} = L_{A^{-1}}$.*

Definition. *Let V and W be vector spaces. We say that V is* ***isomorphic*** *to W if there exists a linear transformation T: V $\rightarrow$ W that is invertible. Such a linear transformation is called an* ***isomorphism*** *from V onto W.*

Theorem 2.19. *Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if dim(V) = dim(W).*

Corollary. *Let V be a vecctor space over F. Then V is isomorphic to F^n if and only if dim(V) = n.*

Theorem 2.20. *Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the function Φ: L(V, W) $\rightarrow M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in$ L(V, W), is an isomorphism.*

Corollary. *Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then L(V, W) is finite-dimensional of dimension mn.*

Definition. *Let β be an ordered basis for an n-dimensional vector space V over the field F. The* ***standard representation of V with respect to*** *β is the function ϕ_β: $V \rightarrow F^n$, defined by $\phi_\beta(x) = [x]_\beta$ for each x $\in$ V.*

Theorem 2.21. *For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism.*

2.5 The Change of Coordinate Matrix

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Theorem 2.22. *Let β and β' be two ordered bases for a finite-dimensional vector space V, and let* $Q = [I_V]_{\beta'}^{\beta}$. *Then*

1. Q is invertible.
2. For any v ∈ V, $[v]_\beta = Q[v]_{\beta'}$.

Definition. *The matrix* $Q = [I_V]_{\beta'}^{\beta}$, *defined in Theorem 2.22 is called a* ***change of coordinate matrix****. Because of part (b) of the theorem, we say that Q* ***changes β'-coordinates into β-coordinates****. Observe that if* $\beta = \{x_1, x_2, ..., x_n\}$ *and* β' = $\{x_1', x_2', ..., x_n'\}$, *then*

$$x_j' = \sum_{i=1}^{n} Q_{ij} x_i \tag{14}$$

for j = 1, 2, ..., n; that is, the jth column of Q is $[x_j']_\beta$.

Theorem 2.23. *Let T be a linear operator on a finite-dimensional vector spsace V, and let β and β′ be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β′-coordinates into β-coordinates. Then*

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \tag{15}$$

Corollary. *Let* $A \in M_{m\times n}(F)$, *and let γ be an ordered basis for* F^n. *Then* $[L_A]_\gamma = Q^{-1}AQ$, *where Q is the n×n matrix whose jth column is the jth vector of γ.*

Definition. *Let A and B be matrices in* $M_{m\times n}(F)$. *We say that B is* ***similar*** *to A if there exists an invertible matrix Q such that* $B = Q^{-1}AQ$.

Aside. ***Determinants:***

Definition. *The* ***determinant*** *of an n×n matrix A having entries from a field F is a scalar in F, denoted by det(A) or —A—, and can be computed in the following manner:*

1. If A is 1×1, then det(A) = A_{11}, the single entry of A.
2. If A is 2×2, then det(A) = $A_{11}A_{22} - A_{12}A_{21}$. For example,

$$det\begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = (-1)(3) - (2)(5) = -13 \tag{16}$$

3. If A is n×n for n > 2, then

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} det(A_{ij}) \tag{17}$$

   (if the determinant is evaluated by the entries of row i of A) or

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} det(A_{ij}) \tag{18}$$

   (if the determinant is evaluated by the entries of column j of A), where A_{ij} is the (n - 1)×(n - 1) matrix obtained by deleting row i and column j from A.

Properties. *(of the Determinant)*

1. If B is a matrix obtained by interchanging any two rows or interchanging any two columns of an n×n matrix A, then det(B) = -det(A).
2. If B is a matrix obtained by multiplying each entry of some row or column of an n×n matrix A by a scalar k. then det(B) = k·det(A).
3. If B is a matrix obtained from an n×n matrix A by adding a multiple of row i to row j or a multiple of column i to column j for i $\neq$ j, then det(B) = det(A).
4. The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, det(I) = 1.
5. If two rows (or columns) of a matrix are identical, then the determinant of the matrix is zero.
6. For any n×n matrices A and B, det(AB) = det(A)· det(B).
7. An n×n matrix A is invertible if and only if det(A) $\neq$ 0. Furthermore, if A is invertible, then det(A^{-1}) = $\frac{1}{det(A)}$.
8. For any n×n matrix A, the determinants of A and A^t are equal.
9. If A and B are similar matrices, then det(A) = det(B).

3 Diagonalization

3.1 Eigenvalues and Eigenvectors

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Definition. *A linear operator T on a finite-dimensional vector space V is called* ***diagonalizable*** *if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called* ***diagonalizable*** *if L_A is diagonalizable.*

Definition. *Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an* ***eigenvector*** *of T if there exists a scalar λ such that T(v) = λv. The scalar λ is called the* ***eigenvalue*** *corresponding to the eigenvector v*
Let A be in $M_{m\times n}(F)$. A nonzero vector $v \in F^n$ is called an ***eigenvector*** *of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ. The scalar λ is called the* ***eigenvalue*** *of A corresponding to the eigenvector v.*

Theorem 5.1. *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, ..., v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.*

Theorem 5.2. *Let $A \in M_{m\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if det(A - λI_n) = 0.*

Definition. *Let $A \in M_{n\times n}(F)$. The polynomial f(t) = det(A - tI_n) is called the* ***characteristic polynomial*** *of A.*

Definition. *Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the* ***characteristic polynomial*** *f(t) of T to be the characteristic polynomial of $A = [T]_\beta$. That is, f(t) = det(A - tI_n).*

Theorem 5.4. *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector v $\in$ V is an eigenvector of T corresponding to λ if and only if v $\neq$ 0 and v $\in$ N(T - λI).*

3.2 Diagonalizability

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Theorem 5.5. *Let T be a linear operator on a vector space V, and let* $\lambda_1, \lambda_2, ..., \lambda_k$ *be distinct eigenvalues of T. If* $v_1, v_2, ..., v_k$ *are eigenvectors of T such that* λ_i *corresponds to* v_i *(*$1 \leq i \leq k$*), then* $\{v_1, v_2, ..., v_k\}$ *is linearly independent.*

Corollary. *Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.*

Definition. *A polynomial f(t) P(F)* ***splits over*** *F if there are scalars* $c, a_1, ..., a_n$ *(not necessarily distinct in F such that*

$$f(t) = c(t - a_1)(t - a_2)...(t - a_n). \tag{19}$$

Theorem 5.6. *The characteristic polynomial of any diagonalizable linear operator splits.*

Definition. *Let* λ *be an eigenvalue of a linear operator or matrix with the characteristic polynomial f(t). The* ***algebraic multiplicity*** *of* λ *is the largest positive integer k for which (t -* $\lambda)^k$ *is a factor of f(t).*

Definition. *Let T be a linear operator on a vector space V, and let* λ *be an eigenvalue of T. Define* E_λ $= \{x \in V : T(x) = \lambda x\} = N(T - \lambda I_V$*. The set* E_λ *is called the* ***eigenspace*** *of T corresponding to the eigenvalue* λ*. Analogously, we define the* ***eigenspace*** *of a square matrix A to be the eigenspace of* L_A*.*

Theorem 5.7. *Let T be a linear operator on a finite-dimensional vector space V, and let* λ *be an eigenvalue of T having multiplicity m. Then* $1 \leq dim(E_\lambda) \leq m$*.*

Lemma. *Let T be a linear operator, and let* $\lambda_1, \lambda_2, ..., \lambda_k$ *be distict eigenvalues of T. For each i = 1, 2, ..., k, let* $v_i \in E_{\lambda_i}$*, the eigenspace corresponding to* λ_i*. If*

$$v_1 + v_2 + ... + v_k = 0. \tag{20}$$

then $v_i = 0$ *for all i.*

Theorem 5.8. *Let T be a linear operator on a vector space V, and let* $\lambda_1, \lambda_2, ..., \lambda_k$ *be distinct eigenvalues of T. For each i = 1, 2, ..., k, let* S_i *be a finite linearly independent subset of the eigenspace* E_{λ_i}*. Then* $S = S_1 \cup S_2 \cup ... \cup S_k$ *is a linearly independent subset of V.*

Theorem 5.9. *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let* $\lambda_1, \lambda_2, ..., \lambda_k$ *be the distinct eigenvalues of T. Then*

1. T is diagonalizable if and only if the multiplicity of λ_i is equal to $dim(E_{\lambda_i})$ for all i.
2. If T is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \cup \beta_2 \cup ... \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T (β is an eigenbasis).

Definition. *Let* $W_1, W_2, ..., W_k$ *be subspaces of a vector space V. We define the* ***sum*** *of these subspaces to be the set*

$$\{v_1 + v_2 + ... + v_k : v_i \in W_i for 1 \leq i \leq k\}, \tag{21}$$

which we denote by $W_1 + W_2 + ... + W_k$ *or* $\sum_{i=1}^{k} W_i$*.*

Definition. *Let* $W_1, W_2, ..., W_k$ *be subspaces of a vector space V. We call V the* ***direct sum*** *of the subspaces* $W_1, W_2, ..., W_k$ *and write* $V = W_1 \bigoplus W_2 \bigoplus ... \bigoplus W_k$*, if*

$$V = \sum_{i=1}^{k} W_i \tag{22}$$

and

$$W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } j \ (1 \leq j \leq k). \tag{23}$$

4 Inner Product Spaces

4.1 Inner Products and Norms

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Definition. *Let V be a vector space over F. An* ***inner product*** *on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted* $\langle x, y\rangle$*, such that for all x, y, and z in V and all c in F, the following hold:*

1. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
2. $\langle cx, y\rangle = c\langle x, y\rangle$
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$, where the bar denotes complex conjugation.
4. $\langle x, x\rangle > 0$ if x $\neq$ 0.

Definition. *Let* $A \in M_{m\times n}(F)$*. We define the* ***conjugate transpose*** *or* ***adjoint*** *of A to be the matrix* A^* *such that* $(A*)_{ij} = \overline{A_{ji}}$ *for all i, j.*

Theorem 6.1. *Let V be an inner product space. Then for x. y.* $z \in V$ *and* $c \in F$*, the following statements are true.*

1. $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
2. $\langle x, cy, \rangle = \bar{c}\langle x, y\rangle$.
3. $\langle x, 0\rangle = \langle 0, x\rangle = 0$.
4. $\langle x, x\rangle = 0$ if and only if x = 0.
5. If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then y = z.

Definition. *Let V be an inner product space. For* $x \in V$*, we define the* ***norm*** *or* ***length*** *of x by* $\|x\| = \sqrt{\langle x, x\rangle}$*.*

Theorem 6.2. *Let V be an inner product space over F. Then for all x,* $y \in V$ *and* $c \in F$*, the following statements are true.*

1. $\|cx\| = |c|\|x\|$
2. $|x| = 0$ if and only if x = 0. In any case, $|x| \geq 0$
3. (Cauchy Schwarz Inequality) $|\langle x, y\rangle| \leq \|x\|\|y\|$
4. (Triangle Inequality) $\|x+y\| \leq \|x\| + \|y\|$

Definition. *Let V be an inner product space. Vectors x and y in V are* ***orthogonal (perpendicular)*** *if* $\langle x, y\rangle = 0$*. A subset S of V is* ***orthogonal*** *if any two distinct vectors in S are orthogonal. A vector x in V is a* ***unit vector*** *if* $\|x\| = 1$*. Finally, a subset S of V is* ***orthonormal*** *if S is orthogonal and consists entirely of unit vectors.*

4.2 The Gram-Schmidt Orthogonalization Process and Orthogonal Complements

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Definition. *Let V be an inner product space. A subset of V is an* ***orthonormal basis*** *for V if it is an ordered basis that is orthonormal.*

Theorem 6.3. *Let V be an inner product space and* $S = \{v_1, v_2, ..., v_k\}$ *be an orthogonal subset of V consisting of nonzero vectors. If* $y \in span(S)$*, then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i \tag{24}$$

Corollary. *If, in addition to the hypotheses of Theorem 6.3, S is orthonormal and* $y \in span(S)$*, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i \tag{25}$$

Corollary. *Let V be an inner product space, and let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linearly independent.*

Theorem 6.4 (Gram-Schmidt Process). *Let V be an inner product space and* $S = \{w_1, w_2, ..., w_n\}$ *be a linearly independent subset of V. Define S' =* $\{v_1, v_2, ..., v_n\}$*, where* $v_1 = w_1$ *and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n \tag{26}$$

Theorem 6.5 . *Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis* β*. Furthermore, if* $\beta = \{v_1, v_2, ..., v_n\}$ *and* $x \in V$*, then*

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i. \tag{27}$$

Corollary. *Let V be a finite-dimensional inner product space with an orthonormal basis* $\beta = \{v_1, v_2, ..., v_n\}$*. Let T be a linear operator on V, and let A =* $[T]_\beta$*. Then for any i and j,* $A_{ij} = \langle T(v_j), v_i \rangle$*.*

Definition. *Let* β *be an orthonormal subset (possibly infinite) of an inner product space V, and let* $x \in V$*. We define the* ***Fourier coefficients*** *of x relative to* β *to be the scalars* $\langle x, y \rangle$*, where* $y \in \beta$*.*

Definition. *Let S be a nonempty subset of an inner product space V. We define* $S^\perp$ *to be the set of all vectors in V that are orthogonal to every vector in S; that is,* $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$*. The set* $S^\perp$ *is called the* ***orthogonal complement*** *of S.*

Theorem 6.6. *Let W be a finite-dimensional subspace of an inner product spaceV, and let* $y \in V$*. Then there exist unique vectors* $u \in W$ *and* $z \in W^\perp$ *such that* $y = u + z$*. Furthermore, if* $\{v_1, v_2, ..., v_k\}$ *is an orthonormal basis for W, then*

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i. \tag{28}$$

Corollary. *In the notation of Theoem 6.6, the vector u is the unique vector in W that is "closest" to y; that is, for any* $x \in W$*,* $\|y - x\| \geq \|y - u\|$*, and this inequality is an equality if and only if* $x = u$*.*

Definition. *The vector u in the corollary is called the* ***orthogonal projection*** *of y on W.*

Theorem 6.7. *Suppose that* $S = \{v_1, v_2, ..., v_k\}$ *is an orthonormal set in an n-dimensional inner product space V. Then*

1. S can be extended to an orthonormal basis $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ for V.
2. If W = span(S), then $S_1 = \{v_{k+1}.v_{k+2}, ..., v_n\}$ is an orthonormal basis for $W^\perp$.
3. If W is any subspace of V, then dim(V) = dim(W) + dim($W^\perp$).